Finite Math - J-term 2017

Lecture Notes - 1/11/2017

Homework

- Section 4.1 - 1, 5, 7, 9, 10, 11, 12, 13, 17, 20, 21, 23, 25, 26, 27, 28, 31, 33, 65, 68, 69, 71, 75, 80

Section 4.1 - Systems of Linear Equations in Two Variables

Suppose we go to a movie theater and there are two packages for discounted tickets:
Package 1: 2 adult tickets and 1 child ticket for \$32
Package 2: 1 adult ticket and 3 child tickets for \$36

Based off of this information, can we figure how much the adult and child ticket discount prices are?

We can! To do this let A stand for the price of the adult ticket and let C stand for the price of the child ticket, then we get the following two equations from the two packages:

$$\begin{array}{rcrcr} 2A & + & C & = & 32 \\ A & + & 3C & = & 36 \end{array}$$

This is a system of two linear equations in two variables. To find the answer, we need to find a pair of numbers (A, C) which satisfy *both* equations simultaneously.

Definition 1 (System of Two Linear Equations in Two Variables)**.** *Given the* linear system

$$\begin{array}{rcrcr} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ (often written as an ordered pair (x_0, y_0)) is a solution *of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the* solution set *for the system. To* solve *a system is to find its solution set.*

There are a few ways we can go about solving this: *graphically*, using *substitution*, and *elimination by addition*.

Solving by Graphing. To solve this problem by graphing, we simply graph the two equations in the system, then find the intersection. Since we're relying on a graph to find this point, we need to check our solution in the equations of the system.

The blue line is the graph of $2A + C = 32$ and the purple line is the graph of $A + 3C = 36$. The red point is the intersection point $(12, 8)$. So the ticket prices are \$12 for an adult ticket and \$8 for a child ticket. Check the point $(12, 8)$ in both equations:

$$2A + C = 2(12) + 8 = 24 + 8 = 32 \quad \checkmark$$

and

$$A + 3C = 12 + 3(8) = 12 + 24 = 36 \quad \checkmark.$$

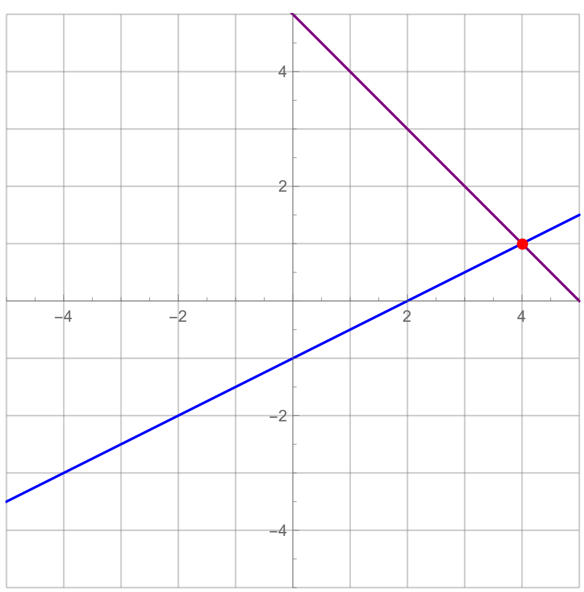
This verifies the solution.

There are actually 3 types of solutions to a system of linear equations

(1) Consider the system

$$\begin{array}{rcrcl} x & - & 2y & = & 2 \\ x & + & y & = & 5 \end{array}$$

If we graph the lines, we get

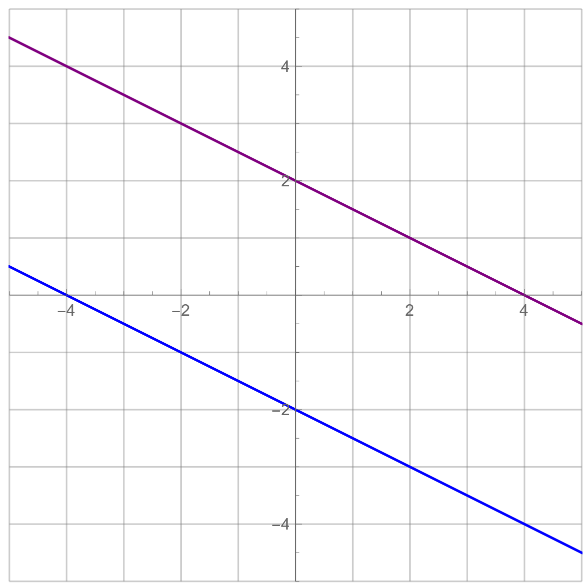


In this case, like before, we see only the *one solution* at $(4, 1)$. (You should check this in the system!)

(2) Consider the system

$$\begin{array}{rcrcr} x & + & 2y & = & 4 \\ 2x & + & 4y & = & 8 \end{array}$$

If we graph the lines, we get

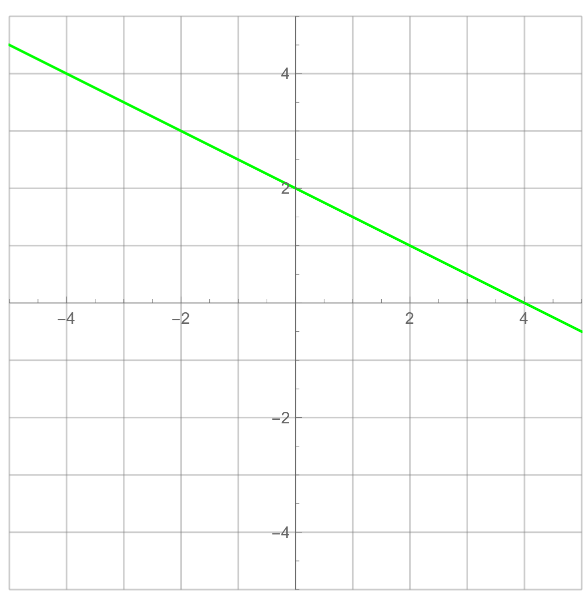


In this case, the lines are parallel and so they never intersect. In this case, there is *no solution*.

(3) Consider the system

$$\begin{array}{rcrcr} 2x & + & 4y & = & 8 \\ x & + & 2y & = & 4 \end{array}$$

If we graph the lines, we get



Here, both of the lines are exactly the same. In this case, there is an infinite number of solutions.

Definition 2. *A system of linear equations is called* consistent *if it has one or more solutions and* inconsistent *if it has no solutions. Further, a consistent system is called* independent *if it has exactly one solution (called the* unique solution*) and is called* dependent *if it has more than one solution. Two systems of equations are called* equivalent *if they have the same solution set.*

Theorem 1. *The linear system*

$$\begin{array}{rcrcl} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

must have

(1) *Exactly one solution (consistentent and independent).*

(2) *No solution (inconsistent).*

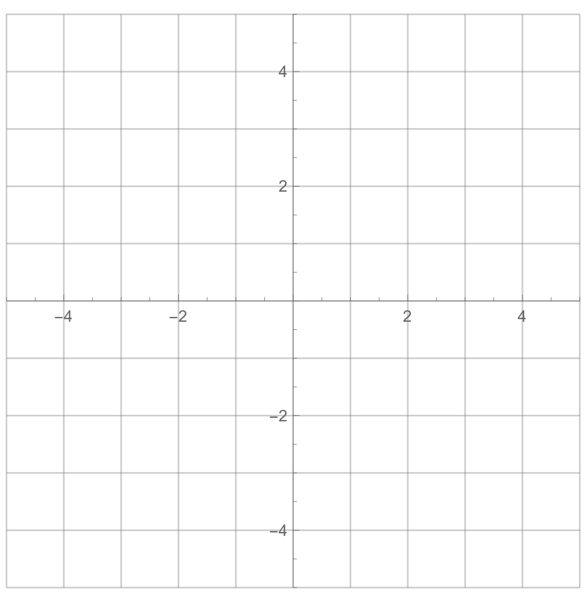
(3) *Infinitely many solutions (consistent and dependent).*

There are no other possibilities.

Example 1. *Solve the following systems of equations using the graphing method. Determine whether there is one solution, no solutions, or infinitely many solutions. If there is one solution, give the solution.*

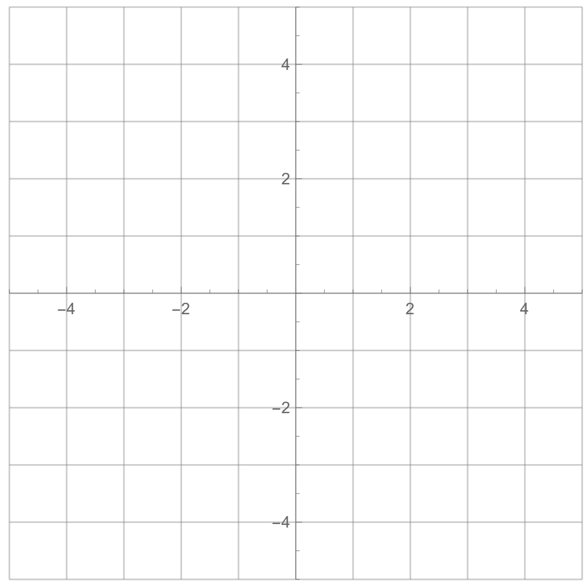
(a)

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ 2x & - & y & = & 5 \end{array}$$



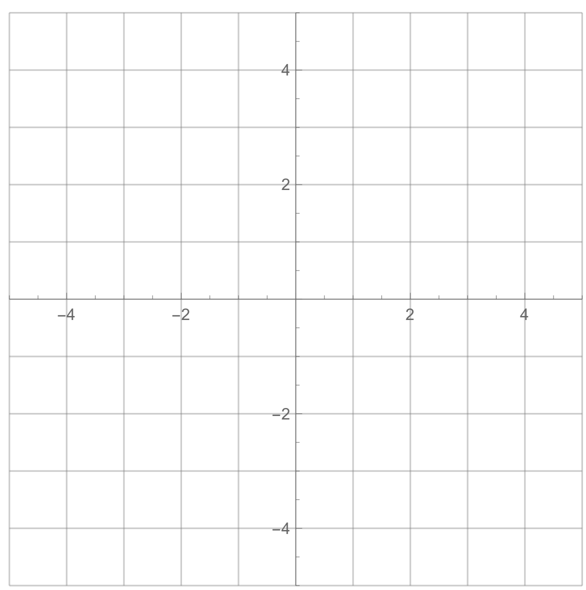
(b)

$$\begin{aligned} 6x - 3y &= 9 \\ 2x - y &= 3 \end{aligned}$$



(c)

$$\begin{aligned} 2x - y &= 4 \\ 6x - 3y &= -18 \end{aligned}$$



Solving by Substitution. When solving a system by substitution, we solve for one of the variables in one of the equations, then plug that variable into the other equation.

Example 2. *Solve the following system using substitution*

$$\begin{array}{rcrcl} 2x & - & y & = & 3 \\ x & + & 2y & = & 14 \end{array}$$

Solution. *Let's solve the first equation for y. To do this, we'll move the y to the right side, and the 3 to the left:*

$$2x - 3 = y$$

Then we plug this into the second equation for y:

$$x + 2(2x - 3) = 14$$

then we solve for x in this

$$x + 4x - 6 = 5x - 6 = 14$$

which gives

$$5x = 20$$

and so

$$x = 4.$$

Then we plug this into the equation we have for y to find that

$$y = 2(4) - 3 = 8 - 3 = 5$$

And so the solutions is $x = 4, y = 5$.

Example 3. *Solve the following system using substitution*

$$\begin{array}{rcrcl} 3x & + & 2y & = & -2 \\ 2x & - & y & = & -6 \end{array}$$

Solution. $x = -2, y = 2$

Solving Using Elimination.

We now turn to a method that, unlike graphing and substitution, is generalizable to systems with more than two variables easily. There are a set of rules to follow when doing this

Theorem 2. *A system of linear equations is transformed into an equivalent system if*

(1) *two equations are interchanged*

(2) *an equation is multiplied by a nonzero constant*

(3) *a constant multiple of one equation is added to another equation.*

Example 4. *Solve the following system using elimination*

$$\begin{array}{rcrcr} 3x & - & 2y & = & 8 \\ 2x & + & 5y & = & -1 \end{array}$$

Solution. *If we subtract the second equation from the first one, we end up with the new system*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ 2x & + & 5y & = & -1 \end{array}$$

Now, we can subtract 2 *times the first equation* ($2x - 14y = 18$) *from the second equation to get*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & 19y & = & -19 \end{array}$$

Now we divide the second equation by 19 *to get*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & y & = & -1 \end{array}$$

and finally, we will add 7 *times the second equation* ($7y = -7$) *to the first equation*

$$\begin{array}{rcrcr} x & & & = & 2 \\ & & y & = & -1 \end{array}$$

This gives the answer of $x = 2, y = -1$.

We could have also used a combination of substitution and elimination above, for example, once we knew that $y = -1$, we could have just plugged that into the first equation, but this solution was a little preview for the later sections.

Example 5. *Solve the system using elimination*

$$\begin{array}{rcrcr} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Solution. $x = 2, y = -1$

We now want to look at the case when the system does not have one unique solution, but is either inconsistent or is consistent but dependent.

Example 6. *Solve the system*

$$\begin{array}{rcrcr} 2x & + & 6y & = & -3 \\ x & + & 3y & = & 2 \end{array}$$

Solution. *We begin by subtracting twice the second equation from the first*

$$\begin{array}{rcrcr} & & 0 & = & -7 \\ x & + & 3y & = & 2 \end{array}$$

The first equation has become $0 = -7$ *which is obviously untrue. This is an example of what happens when the system is* inconsistent.

Example 7. *Solve the system*

$$\begin{array}{rcrcr} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Solution. *First, let's multiply the first equation by 2 to get rid of the fraction*

$$\begin{array}{rcrcr} 2x & - & y & = & 8 \\ -2x & + & y & = & -8 \end{array}$$

Now, if we add the two equations together, we get

$$0 = 0$$

which is always true. This means that the two equations are the same equation, just one is (maybe) multiplied by a constant. This is a consistent but dependent *system of equations. If we let $x = k$, where k is any real number, then we get that $y = 2k-8$. So, for any k, $(k, 2k-8)$ is a solution. In this case, the variable k is called a* parameter*.*

Example 8. *Solve the systems*

(a)

$$\begin{array}{rcrcr} 5x & + & 4y & = & 4 \\ 10x & + & 8y & = & 4 \end{array}$$

(b)

$$\begin{array}{rcrcr} 6x & - & 5y & = & 10 \\ -12x & + & 10y & = & -20 \end{array}$$

Applications.

There are a variety of applications of systems of equations. For a simple example, consider the following

Example 9. *Dennis wants to use cottage cheese and yogurt to increase the amount of protein and calcium in his daily diet. An ounce of cottage cheese contains 3 grams of protein and 15 milligrams of calcium. An ounce of yogurt contains 1 gram of protein and 41 milligrams of calcium. How many ounces of cottage cheese and yogurt should Dennis eat each day to provide exactly 62 grams of protein and 760 milligrams of calcium?*

Solution. *We begin by setting up an equation for the amount of protein consumed and another equation for the amount of calcium consumed. Suppose Dennis eats x ounces of cottage cheese and y ounces of yogurt. Since 1 ounce of cottage cheese contains 3 grams of protein we know that x ounces of cottage cheese will contain $3x$ grams of* *protein; likewise y ounces of yogurt contains y grams of protein. Since Dennis wants to consume a total of 62 grams of protein, we get the equation*

$$3x + y = 62$$

We similarly set up an equation for milligrams of calcium consumed: x ounces of cottage cheese has $15x$ milligrams of calcium and y ounces of yogurt has 41 milligrams of calcium, and Dennis wants to eat exactly 760 milligrams of calcium, so

$$15x + 41y = 760$$

Thus we have a system of equations to solve which will tell us exactly how much cottage cheese and yogurt Dennis should eat

$$\begin{array}{rcrcl} 3x & + & y & = & 62 \\ 15x & + & 41y & = & 760 \end{array}$$

If we multiply the top equation by -5 then add the two equations together, we get

$$36y = 450$$

giving

$$y = 12.5$$

Plugging this into the first equation gives us

$$3x + 12.5 = 62 \iff 3x = 49.5 \iff x = 16.5.$$

So, Dennis should eat 16.5 grams of cottage cheese and 12.5 grams of yogurt each day to reach his target.

Example 10. *A fruit grower uses two types of fertilizer in an orange grove, brand A and brand B. Each bag of brand A contains 8 pounds of nitrogen and 4 pounds of phosphoric acid. Each bag of brand B contains 7 pounds of nitrogen and 6 pounds of phosphoric acid. Tests indicate that the grove needs 720 pounds of nitrogen and 500 pounds of phosphoric acid. How many bags of each brand should be used to provide the required amounts of nitrogen and phosphoric acid?*

Solution. 41 *bags of brand A and* 56 *bags of brand B.*

In a free market economy, the price of a product is determined by the relationship between supply and demand. Suppliers are more willing to supply product when the price is high and consumers have a higher demand for a product when the price is low. In a free competitive market, the price tends to move toward an *equilibrium price*, where the supply and demand are equal. The supply and demand at that price is called the *equilibrium quantity*. Graphically, this will be the intersection of the supply and demand curves (which we will assume to be lines here).

Example 11. *At a price of* \$1.88 *per pound, the supply for cherries in a large city is* 16,000 *pounds and the demand is* 10,600 *pounds. When the price drops to* \$1.46 *per pound, the supply decreases to* 10,000 *pounds and the demand increases to* 12,700 *pounds. Assume that the price-supply and price-demand equations are linear.*

(a) *Find the price-supply equation.*

(b) *Find the price-demand equation.*

(c) *Find the supply and demand at a price of* \$2.09 *per pound.*

(d) *Find the supply and demand at a price of* \$1.32 *per pound.*

(e) *Find the equilibrium price and equilibrium demand.*

Solution. *Let x be the quantity in thousands of pounds and let p be the price. We will write points as (x, p).*

(a) *When the price is* \$1.88 *the supply is* 16,000*, so we have the point* $(16, 1.88)$*; and when the price is* \$1.46 *the supply is* 10,000*, giving the point* $(10, 1.46)$*. Using the point-slope formula for a line, we can find the equation for the supply curve:*

$$p - 1.88 = \frac{1.46 - 1.88}{10 - 16}(x - 16) = 0.07(x - 16)$$

or,

$$p = 0.07x + 0.76.$$

(b) *When the price is* \$1.88 *the demand is* 10,600*, so we have the point* $(10.6, 1.88)$*; and when the price is* \$1.46 *the demand is* 12,700*, giving the point* $(12.7, 1.46)$*. Using the point-slope formula for a line, we can find the equation for the demand curve:*

$$p - 1.88 = \frac{1.46 - 1.88}{12.7 - 10.6}(x - 10.6) = -0.2(x - 10.6)$$

or,

$$p = -0.2x + 4.$$

(c) *Here, we just plug in $p = 2.09$ to our equations and find x:*
The supply will be

$$2.09 = 0.07x + 0.76 \quad \Longrightarrow \quad 0.07x = 1.33$$

so the supply is

$$x = 19$$

or 19,000 *pounds.*
The demand will be

$$2.09 = -0.2x + 4 \quad \Longrightarrow \quad 0.2x = 1.91$$

so the demand is

$$x = 9.55$$

or $9,550$ *pounds.*

(d) *We do the same thing as in the previous part, just plugging in* $p = 1.32$ *instead. The supply will be*

$$1.32 = 0.07x + 0.76 \quad \Longrightarrow \quad 0.07x = 0.56$$

so the supply is

$$x = 8$$

or $8,000$ *pounds. Notice that the supply went down as the price decreased. The demand will be*

$$1.32 = -0.2x + 4 \quad \Longrightarrow \quad 0.2x = 2.68$$

so the demand is

$$x = 13.4$$

or $13,400$ *pounds. Notice that the demand went up with the decreased price.*

(e) *To find the equilibrium, we need to solve the system of equations determined by the supply and demand equations. Since both are already solved for* p*, we will use the substitution method to get*

$$0.07x + 0.76 = -0.2x + 4 \quad \Longrightarrow \quad 0.27x = 3.24$$

giving

$$x = 12$$

Thus the equilibrium quantity is $12,000$ *pounds, and the equilibrium price is (plug* x *into either equation)*

$$p = -0.2(12) + 4 = 1.60$$

that is, $\$1.60$ *per pound.*